Journal of Experimental and Theoretical Physics, Vol. 91, No. 5, 2000, pp. 894–898.
Translated from Zhurnal Éksperimental'noĭ i Teoreticheskoĭ Fiziki, Vol. 118, No. 5, 2000, pp. 1036–1040.
Original Russian Text Copyright © 2000 by Gorbachev, Trubilko.

ATOMS, SPECTRA, RADIATION

Quantum Teleportation of an Einstein–Podolsy–Rosen Pair Using an Entangled Three-Particle State

V. N. Gorbachev* and A. I. Trubilko**

St. Petersburg Institute of Moscow State University of Printing, St. Petersburg, 190000 Russia
**e-mail: vn@vg3025.spb.edu*
***e-mail: tai@at3024.spb.edu*
Received October 13, 1999

Abstract—Quantum teleportation of an Einstein–Podolsky–Rosen pair using maximally entangled triplets to two receivers is studied. The projection basis for combined three-particle measurements, from the results of which the unknown state can be reconstructed, is found. The basis contains states where only two of the three particles are maximally entangled. *© 2000 MAIK "Nauka/Interperiodica".*

1. INTRODUCTION

Quantum teleportation was proposed in [1]. It enables the sender *A* (conventionally named Alice) to transmit an unknown state to the user *B* (Bob) located in a different location of space. Teleportation of a state of a two-level quantum system or a qubit (quantum bit) requires an Einstein–Podolsky–Rosen (EPR) pair and a classical communication channel, along which the result of combined measurements performed in the Bell basis is transmitted. Various variants of this process, which employ two-particle entangled states, are examined in [2]. The case of quantum teleportation of a polarized photon and single-mode coherent light has been demonstrated experimentally in [3, 4]. In [4] light from an optical parametric amplifier was used as the source of the EPR pair, but the physical nature of the particles can be arbitrary. Thus, in [5] an ensemble of EPR correlated atoms was obtained. Such two-particle entangled states can be used in schemes for interspatial teleportation [6], where a quantum state is transferred between particles of a different physical nature [7].

The quantum teleportation procedure can be regarded as a computational process [8], corresponding to a network consisting of rotation type operations, logical elements c-NOT (controlled NOT), and other components. Such schemes are interesting because the components can be implemented using physical systems of various nature, for example, optical, such as a half-transmitting mirror or a polarized divider [9].

In the present paper we consider A variant of quantum teleportation of an entangled pair to two receivers *B* and *C* (Clair) using a three-particle entangled state of the type GHZ (Greenberger–Horne–Zeilinger). We note that the GHZ triplet has been realized experimentally in [10].

The main problem arising here concerns the form of the three-particle projection basis required for the combined measurements. The basis found, in contrast to the variants of quantum teleportation of one qubit, is not maximally entangled. It contains a state with two maximally entangled particles. Two receivers can reconstruct the unknown wave function of the EPR pair from the results of combined measurements in this basis, but neither receiver can do so separately. As shown in [11], in such a scheme with a GHZ triplet, where one qubit participates instead of an EPR pair, the unknown single-particle state can be reconstructed by only one receiver. The results obtained were extended to the case of the quantum teleportation of an *N*-particle entangled state of the EPR type. The basic features of the quantum teleportation process for one particle are presented in Section 2. The types of initial states of the entangled pair and the triplet are determined in Section 3, and the projection operators for combined measurements for them are constructed in Section 4. In Section 5 a protocol and a quantum teleportation scheme for an EPR pair are presented and the case of an *N*-particle entangled state of the EPR type is studied.

2. TELEPORTATION

According to [1], the teleportation of an unknown state of a quantum system by a sender *A* to a receiver *B* located at a different point in space involves the following basic aspects.

Let *A* possess a physical system, for example, a two-level or qubit system, in the unknown state

$$|\psi_1\rangle = \alpha|0\rangle + \beta|1\rangle, \quad (1)$$

where $|\alpha|^2 + |\beta|^2 = 1$. An EPR pair in the maximally entangled state $|\Psi_{23}\rangle = (|01\rangle + |10\rangle)/\sqrt{2}$ is distributed between *A* and *B*, so that qubit 2 is located at *A* and qubit 3 is located at *B*. First, *A* performs combined

measurements on qubits 1 and 2 in the Bell basis, consisting of the for projecors $\prod_k = |\pi_k\rangle\langle\pi_k|$, $k = 1, \ldots, 4$, $\sum_k \prod_\kappa = 1$, $|\pi_1\rangle = |\Phi_{12}^+\rangle$, $|\pi_2\rangle = |\Phi_{12}^-\rangle$, $|\pi_3\rangle = |\Psi_{12}^+\rangle$, $|\pi_4\rangle = |\Psi_{12}^-\rangle$, where the Bell states represent the maximally entangled two-particle states

$$|\Phi^\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle), \quad (2)$$

$$|\Psi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle). \quad (3)$$

As result of combined measurements on the particles 1 and 2, the density matrix of the entire system $\rho = |\psi_1\rangle\langle\psi_1| \otimes |\Psi_{23}\rangle\langle\Psi_{23}|$, determined in the three-particle Hilbert space $H_1 \otimes H_2 \otimes H_3$, projects onto one of the four Bell states. The main circumstance ensuring the success of the procedure is that the probability of the kth outcome $\Pr(k) = \mathrm{Sp}\{\prod_k \rho \prod_k^\dagger\} = 1/4$ does not depend on α and β, and the reduced density matrix $\rho_3(k) = \mathrm{Sp}_{12}\{\prod_k \rho \prod_k^\dagger\}$ of qubit 3 is related with the unknown matrix of qubit one by the unitary transformation U_k:

$$\rho_3(k) = U_k \tilde{\rho}_1 U_k^\dagger, \quad (4)$$

where $\tilde{\rho}_1$ is the density matrix in H_3, corresponding to $\rho_1 = |\psi_1\rangle\langle\psi_1|$, and U_k is a set of unitary operators consisting of the Pauli matrices $U_1 = \sigma_x$, $U_2 = -i\sigma_y$, $U_3 = 1$, $U_4 = \sigma_z$.

According to the protocol of [1], the sender A sends the kth outcome of the measurement to the receiver B, who performs on his qubit 3 from the EPR pair one of four operations U_k, according to the message received. Ultimately, a qubit in the state $|\psi_1\rangle$ arises at B and teleportation has been successfully completed.

3. RESOURCES

Teleportation of an EPR pair requires a maximally entangled triplet of particles. On the basis of this assertion we shall examine the possible types of initial states.

The entangled pair can be described by a wave function of the form

$$|\Psi_{12}\rangle = \alpha|00\rangle + \beta|11\rangle, \quad (5)$$

where $|\alpha|^2 + |\beta|^2 = 1$, or

$$|\Psi_{\mathrm{EPR}}\rangle = \alpha|01\rangle + \beta|10\rangle, \quad (6)$$

which corresponds to the case of an EPR pair. Eight states with three maximally entangled particles can be indicated, for example,

$$\begin{aligned} &(|000\rangle \pm |111\rangle)/\sqrt{2}, \quad (|001\rangle \pm |110\rangle)/\sqrt{2}, \\ &(|010\rangle \pm |101\rangle)/\sqrt{2}, \quad (|100\rangle \pm |011\rangle)/\sqrt{2}. \end{aligned} \quad (7)$$

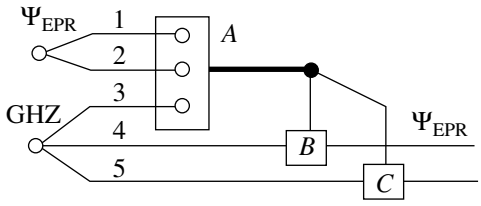

Fig. 1. Network for teleportation of an EPR pair using a GHZ triplet.

Without loss of generality we choose from the set of possible initial states presented above the state (6) and a GHZ triplet:

$$|\Psi_{\mathrm{GHZ}}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle). \quad (8)$$

Then the initial wave function of the entire system is a product

$$|\Psi\rangle = |\Psi_{\mathrm{EPR}}\rangle \otimes |\Psi_{\mathrm{GHZ}}\rangle. \quad (9)$$

Figure 1 displays a scheme for teleportation of an EPR pair, formed by particles 1 and 2, using an entangled triplet of particles 3, 4, and 5. The GHZ triplet is distributed between the sender A and the receivers B and C, to whom A sends the results of combined measurements on particles 1, 2, and 3. Eight projection operators $\prod_k$, forming a complete basis in which the wave function $|\Psi\rangle$ can be expanded, are required in order to perform combined measurements. The choice of such a basis is the main ingredient for solving the problem.

4. PROJECTION BASIS

It could be inferred that the set of projection operators $\prod_k$ will consist of the maximally entangled states (7). We denote this basis by $\pi_{(123)}$. However, a direct calculation shows that this is not so. The problem is that the projections of the initial wave function $|\Psi\rangle$ on the four vectors $(|010\rangle \pm |101\rangle)/\sqrt{2}$ and $(|100\rangle \pm |011\rangle)/\sqrt{2}$ from $\pi_{(123)}$ are 0. It is impossible to reconstruct the states of an EPR pair from these measurements. Consequently, here, in contrast to teleportation of one particle, the maximally entangled basis does not solve the problem.

In the present case a basis with two maximally entangled particles 1, 3 or 1, 2 is suitable. However, the presence of an entangled pair is only a necessary condition. To study the possible realizations of the operators $\prod_k$ we introduce a classification where one of the indicators will be the number of maximally entangled particles (two or three). Since all complete sets are related with one another by a unitary transformation, one can be chosen as the initial set. Let the initial basis be π_{123}:

$$|\pi_{123}\rangle = |ijk\rangle, \quad i, j, k = 0, 1, \quad (10)$$

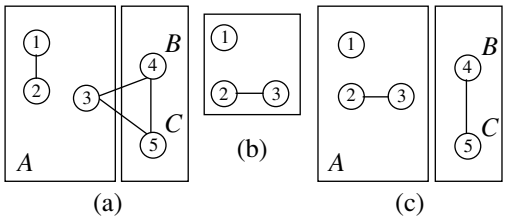

Fig. 2. Structure of the wave functions: (a) initial state with entangled particles 1, 2 from an EPR pair and 3, 4, 5 from a GHZ triplet; (b) projection basis $\pi_{1(23)}(4)$; (c) state after a measurement.

where each of the eight components describes a state with three independent particles. Any component of a different basis can be represented as a linear combination of $s \le 8$ vectors from π_{123}. In what follows, we shall assume the number s to be the same within a single basis and we shall use it for classification. This gives a basis $\pi_{1(23)}(s)$ in which particles 2 and 3 are maximally entangled. For $s = 2$ it has the form

$$|\pi_{1(23)}(2)\rangle = \{|i\rangle|\Phi^{\pm}_{23}\rangle;\ |i\rangle|\Phi^{\pm}_{23}\rangle\}, \quad i = 0, 1, \qquad (11)$$

where each of eight vectors, for example, $|0\rangle|\Phi^{\pm}_{23}\rangle = (|000\rangle \pm |011\rangle)/\sqrt{2}$, is represented by two elements from π_{123}. For the case $s = 4$

$$|\pi_{1(23)}(4)\rangle = \{|\pi^{\pm}_1\rangle|\Phi^{\pm}_{23}\rangle;\ |\pi^{\pm}_1\rangle|\Phi^{\pm}_{23}\rangle\}, \qquad (12)$$

where a pair of vectors forming a single-particle basis has the form

$$|\pi^{\pm}_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm \exp(i\varphi)|1\rangle). \qquad (13)$$

Any of the sets presented above is complete and orthonormal, but $\pi_{1(23)}(2)$ does not solve the problem. The reason is that the probabilities of the outcomes of measurements in this basis depend on the parameters of the teleportation wave function $|\Psi_{\text{EPR}}\rangle$: $\Pr(k) = |\alpha|^2/4$, $|\beta|^2/4$. Consequently, the unitary transformation relating the reduced density matrix of particles 3 and 4 with the state of the EPR pair will depend on α and β, i.e., on the unknown state. We note that for the basis $\pi_{(12)3}(4)$ their arises a situation, just as for $\pi_{(123)}(2)$, where half of the expansion coefficients are zero.

Two bases with $s = 4$ are suitable for teleportation of an EPR pair: $\pi_{1(23)}(4)$ or $\pi_{(13)2}(4)$, where the particles 2, 3 or 1, 3 are entangled. The structure of the initial state, the projection basis $\pi_{1(23)}(4)$, and the wave function of the entire system after the measurements are shown in Fig. 2.

5. TELEPORTATION OF AN EPR PAIR

The expansion of the wave function of the initial state in the basis $\pi_{1(23)}$, determined according to Eq. (12), where we set the phase $\varphi = 0$, has the form

$$\begin{aligned}|\Psi\rangle = {} & |\pi^{+}_1\rangle|\Phi^{+}_{23}\rangle|1\rangle + |\pi^{+}_1\rangle|\Phi^{-}_{23}\rangle|2\rangle \\ & + |\pi^{-}_1\rangle|\Phi^{+}_{23}\rangle|3\rangle + |\pi^{-}_1\rangle|\Phi^{-}_{23}\rangle|4\rangle \\ & + |\pi^{+}_1\rangle|\Phi^{+}_{23}\rangle|5\rangle + |\pi^{+}_1\rangle|\Phi^{-}_{23}\rangle|6\rangle \\ & + |\pi^{-}_1\rangle|\Phi^{+}_{23}\rangle|7\rangle + |\pi^{-}_1\rangle|\Phi^{-}_{23}\rangle|8\rangle,\end{aligned} \qquad (14)$$

where

$$\begin{aligned}|1, 2\rangle &= \beta|00\rangle \pm \alpha|11\rangle, \\ |3, 4\rangle &= -(\beta|00\rangle \mp \alpha|11\rangle), \\ |5, 6\rangle &= \beta|11\rangle \pm \alpha|00\rangle, \\ |7, 8\rangle &= -(\beta|11\rangle \mp \alpha|00\rangle).\end{aligned} \qquad (15)$$

Equations (15) mean that for measurements in the chosen basis the reduced density matrix $\rho_{45}(k) = \text{Sp}_{123}\{\Pi_k|\Psi\rangle\langle\Psi|\Pi_k\}$ of particles 4 and 5 will be related with the density matrix of the EPR pair by a unitary transformation

$$\rho_{45}(k) = U_k\tilde{\rho}_{EPR}U^{\dagger}_k, \quad k = 1, \ldots, 8, \qquad (16)$$

where $\tilde{\rho}_{\text{EPR}} = |\tilde{\Psi}_{\text{EPR}}\rangle\langle\tilde{\Psi}_{\text{EPR}}|$, $\tilde{\Psi}_{\text{EPR}}$ is a wave function from the Hilbert space $H_4 \otimes H_5$, corresponding to Ψ_{EPR}. The following expressions are valid for the unitary operator from Eq. (16): $U_1 = \sigma_{x4} \otimes I_5$, $U_2 = -U_3 = i\sigma_{y4} \otimes I_5$, $U_4 = -U_1$, $U_5 = I_4 \otimes \sigma_{x5}$, $U_6 = -U_7 = I_4 \otimes (-i\sigma_{y5})$, $U_8 = -U_5$. Here the Pauli operators $\sigma_{\gamma j}$ ($\gamma = x, y, z$) and the identity operators I_j act on the variables of the particle $j = 4, 5$.

In this scheme the teleportation of an EPR pair is accomplished according to the following protocol.

1. The sender A performs eight measurements in the basis $\pi_{1(23)}(4)$ on particles 1, 2, and 3, whose outcomes she transmits to B and C.

2. For the outcomes $k = 1$–4 the receivers B uses the unitary transformations σ_x, $i\sigma_y$, $-i\sigma_y$, $-\sigma_x$ on his particle. Ultimately, an EPR pair in the state Ψ_{EPR} arises at B and C.

3. For the outcomes $k = 5, 6$ the receiver C applies the unitary operations σ_x, $-i\sigma_y$, $i\sigma_y$, $-\sigma_x$ to her particle in order to reconstruct the state of the EPR pair.

In the protocol presented above, in half the cases only one receiver applies a unitary operation to her particle; the other receiver at the same time "operates" with the identity operator on her particle. This variant is not unique. The problem is that the initial states can be reconstructed by different methods. For example, the vector $|2\rangle$ from Eq. (15) can be obtained in two ways:

$$\beta|00\rangle - \alpha|11\rangle = i\sigma_{y4} \otimes I_5|\tilde{\Psi}_{\mathrm{EPR}}\rangle = \sigma_{x4} \otimes \sigma_{z5}|\tilde{\Psi}_{\mathrm{EPR}}\rangle. \quad (17)$$

The expression (17) means that for the outcome of a measurement, where $k = 2$, both receivers, *B* and *C*, must simultaneously operate on their particles, just as in the protocol presented, applying the operations σ_{x4} and σ_{z5} instead of $i\sigma_{y4}$ and the identity operation. However, these differences do not change the final results. The general feature here is the presence of two receivers, neither of which can solve the problem separately.

Figure 3 displays a network that simulates the teleportation of an EPR pair. It was constructed similarly to the case of one particle [8, 9, 12] and contains the c-NOT operation and the Hadamard transformation *H*. At the output of the EPR block particles 1 and 2 are in the entangled state Ψ_{EPR} after the c-NOT operation C_{12}, acting on qubits 1 and 2. The operation C_{12} does not change the state of qubit 1 hand to flips qubit 2 (the target) only when 1 is in a state corresponding to the logical 1. In the GHZ a maximally entangled GHZ triplet is produced by means of the Hadamard transformation *H*, operating qubit 3, ($(H|0\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$, $H|1\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$), and two c-NOT operations C_{34} and C_{35}. At the output of the entire scheme the qubits 4 and 5 are in the entangled state Ψ_{EPR}, which does not depend on the states of the other qubits. We note that here, together with Ψ_{EPR}, a pair with the wave function (5) can be used.

The procedure examined above can be extended to the case of the teleportation of an *N*-particle entangled state of the EPR type

$$|\Psi_N\rangle = \alpha|0\rangle^N + \beta|1\rangle^N \quad (18)$$

using $N + 1$ qubits in a maximally entangled state of the GHZ type

$$|\Psi_{(N+1)}\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{N+1} + |1\rangle^{N+1}), \quad (19)$$

where $|i\rangle^N = |i\rangle \otimes \ldots |i\rangle$, $i = 0, 1$. This scheme involves $2N + 1$ particles, the sender *A*, and *N* receivers, between which $N + 1$ particles from Eqs. (19) are distributed. The initial wave function, which is determined in the Hilbert space $H_1 \otimes \ldots H_{2N+1}$, is a product $|\Psi\rangle = |\Psi_N\rangle \otimes |\Psi_{(N+1)}\rangle$.

To perform combined measurements on $N + 1$ particles 1, 2, …, *N*, $N + 1$ it is necessary to have 2^{N+1} projection operators, forming a complete set with a maximally entangled pair *M*, $N + 1$, $M = 1, \ldots, N$. For $M = N$ such a basis can be represented in the form

$$\pi_{1,\ldots,N-1(N,N+1)}(s) = \{|\pi_{1,\ldots,N-1}\rangle \otimes |\Phi^{\pm}_{N,N+1}\rangle; |\pi_{1,\ldots,N-1}\rangle \otimes |\Phi^{\pm}_{N,N+1}\rangle\}, \quad (20)$$

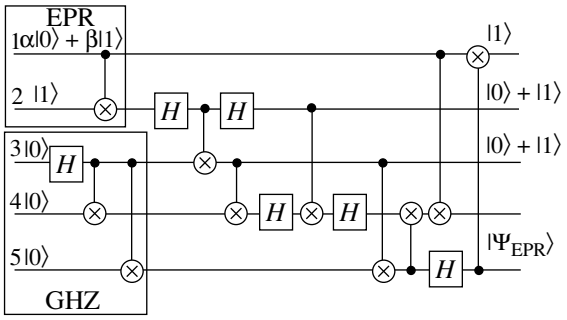

Fig. 3. Network for teleportation of an EPR pair.

where the particles of *N*, $N + 1$ are entangled, and $|\pi_{1,\ldots,N-1}\rangle$ are vectors from $H_1 \otimes \ldots H_{N-1}$. As noted above, the presence of an entangled pair is only a necessary condition for a projection basis. The value of the parameter *s*, whose value and the form of the vectors $\pi_{1,\ldots,N-1}$ will be determined below, expanding the wave function of the entire system in the basis (20), will be a sufficient condition. The expansion has the form

$$|\Psi\rangle = \{P_{N-1}\alpha|0\rangle^N \pm Q_{N-1}\beta|1\rangle^N\} \times |\pi_{1,\ldots,N-1}\rangle|\Phi^{\pm}_{N,N+1}\rangle + \{P_{N-1}\alpha|1\rangle^N \pm Q_{N-1}\beta|0\rangle^N\} \times |\pi_{1,\ldots,N-1}\rangle|\Phi^{\pm}_{N,N+1}\rangle, \quad (21)$$

where

$$P_{N-1} = \langle\pi_{1,\ldots,N-1}|0\rangle^{N-1}, \quad Q_{N-1} = \langle\pi_{1,\ldots,N-1}|1\rangle^{N-1}. \quad (22)$$

Teleportation requires that

$$P_{N-1} \neq 0, \quad Q_{N-1} \neq 0. \quad (23)$$

This means that all coefficients in the expansion must contain a linear combination of two terms $\alpha|i\rangle^N$ and $\beta|j\rangle^N$, $i \neq j = 0, 1$, which can be reconstructed from Eq. (18) by a unitary transformation that does not depend on α and β, i.e., on the parameters of the unknown state. The set of vectors

$$|\pi_{1,\ldots,N-1}\rangle = \{|\pi_1^{\pm}\rangle^{N-1}\}, \quad (24)$$

where $\pi_1^{\pm}$ is the single-particle basis determined according to Eq. (13), satisfies the conditions (23). This can be shown by noting that the set (24) consists of 2^{N-1} elements, each of which always contains a pair of terms $|i\rangle^{N-1}$, $i = 0, 1$.

For the basis found from Eqs. (20) and (24), the parameter *s* will have the value $s = 2^N$. We note that the case $s < 2^N$, where bases containing more than a pair of entangled particles (two pairs or a triplet) arise, does not solve the problem. The following assertion can serve as the final result. Teleportation of an *N*-particle entangled state of the EPR type, using maximally

entangled $N + 1$ particles for combined measurements, requires a projection basis containing one maximally entangled pair. Each element of the basis must consist of 2^N vectors corresponding to the states of $N + 1$ independent particles.

ACKNOWLEDGMENTS

We thank the Delzell Foundation for partial support of this work.

REFERENCES

1. C. H. Bennett, G. Brasard, C. Crepeau, *et al.*, Phys. Rev. Lett. **70**, 1895 (1993).
2. L. Vaidman, Phys. Rev. A **49**, 1473 (1994); S. N. Molotkov, Pis'ma Zh. Éksp. Teor. Fiz. **68**, 248 (1998) [JETP Lett. **68**, 263 (1998)]; P. van Loock, S. L. Braunstein, and H. J. Kimble, quant-ph/9902030.
3. D. Bouwmeester, J.-W. Pan, M. Mattle, *et al.*, Nature **390**, 575 (1997); D. Boschi, S. Branca, F. De Martini, *et al.*, Phys. Rev. Lett. **80**, 1121 (1998).
4. A. Furusawa, J. L. Sorensen, S. L. Braunstein, *et al.*, Science **282**, 706 (1998).
5. A. Kusmich, K. Molmer, and E. S. Polzik, Phys. Rev. Lett. **79**, 4782 (1997); J. L. Sorensen, J. Hald, and E. S. Polzik, Phys. Rev. Lett. **80**, 3487 (1998).
6. C. S. Maierle, D. A. Lidar, and R. A. Harris, quant-ph/9807020.
7. A. S. Parkins and H. J. Kimble, quant-ph/9904054.
8. G. Brassard, quant-ph/9605035.
9. C. A. Adami and N. J. Cerf, quant-ph/9806048.
10. D. Bouwmeester, J. Pan, M. Daniell, *et al.*, quant-ph/9810035; R. J. Nelson, D. G. Cory, and S. Lloyd, quant-ph/9905028.
11. A. Karlson and M. Bourennane, Phys. Rev. A **58**, 4394 (1998).
12. S. L. Braunstein, Phys. Rev. A **53**, 1900 (1996).

Translation was provided by AIP